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Constraint Violation Stabilization Using Input–Output Feedback Linearization in Multibody Dynamic Analysis

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A constraint violation stabilization technique for solving differential algebraic equations (DAE) of multibody dynamic systems is presented. The technique, based on the input–output feedback linearization, is employed to transform the nonlinear DAE into a set of linear equations. On reaching the input–output linear relationship with proven stable zero dynamics, a robust control design is adopted to construct constraint forces that can be used to effectively correct the errors accumulated in the constraint equations during the process of time integration. In the present development, if the pole placement method is used in control design of the resulting linear differential equations, constraint forces based on Baumgarte's constraint violation stabilization technique are recovered. On the other hand, if variable structure control design is adopted, a new method in calculating constraint forces is obtained. Two numerical examples are used to demonstrate the effectiveness of stabilizing the constraint violation by using the proposed technique.

I. Introduction

COMPUTER simulation of multibody dynamic systems is becoming increasingly important for its capability to design and analyze engineering problems such as robotic arm manipulators, ground vehicles, and spacecraft. In general, the dynamic equations of multibody dynamic (MBD) systems can be derived and expressed in various forms depending on the choice of reference frames or coordinate systems used to describe the bodies. As a result, several MBD formulations, which differ primarily in their incorporation of constraints and their resulting system topologies, have been developed.[1–5] Hence, the efficiency and the reliability of these formulations have been strongly affected by the computational techniques employed to solve the governing equations of motion, as well as the accurate treatment of a large number of holonomic and nonholonomic constraints.

Because of the importance of treating the constraints accurately and reliably in the simulation of MBD systems, several computational procedures have been proposed in the past two decades. These include the singular decomposition technique by Walton and Steeves,[6] the stabilization technique by Baumgarte,[7,8] the coordinate partitioning technique by Wehage and Haug,[9] the differential algebraic approach by Gear[10] and Petzold,[11] the penalty method by Orlendea et al.[12] and Lötstedt,[13] the penalty staggered stabilized procedure by Park and Chiou,[14] and the gradient feedback by Yoon et al.[15]

Among the techniques cited, Baumgarte's technique[7,8] is considered the most widely used technique for handling both holonomic and nonholonomic constraints. Baumgarte's technique simply modified the original constraint equations to form a set of relaxation differential constraint equations to stabilize the constraint violations. Note that his output-feedback-like control scheme did not consider the internal dynamics of the differential algebraic equations (DAE), which may cause the overall system to become unstable. In this regard, input–output feedback linearization is proposed to study the internal dynamics of the DAE. Input–output feedback linearization has been successfully used to address some practical nonlinear control problems, including the control of high-performance aircraft,[16] the control of robot arms with elastic joints,[17] and the control of constrained dynamic systems[18] for over a decade. The central feature of this method is transformation of a set of nonlinear dynamic equations into equivalent systems of linear form so that linear control design methods can be applied.[19] In this paper, a new constraint violation stabilization technique that is based on input–output feedback linearization and variable structure control design is reported. The resulting constraint forces are capable of improving the constraint violation in comparison with Baumgarte's constraint violation stabilization technique. The stability of the internal dynamics of a general multibody system is discussed in the Appendix.

To this end, the paper is organized as follows. Section II presents a review of the Lagrangian method for formulating the equations of motion with constraints. Section III presents a review of the formulation framework of input–output feedback linearization and applies this technique to transform DAEs into a set of linear equations. Section IV presents different types of control strategies to obtain various constraint forces. Finally, numerical experiments are reported in Sec. V to illustrate the effectiveness of using the proposed technique to stabilize the constraint violation.

II. Equations of Motion with Holonomic Constraints

The Lagrangian λ method is the most popular technique for deriving the equations of motion with constraints, including both configuration (holonomic) and motion (nonholonomic) constraints. To focus our subsequent discussion, we specialize the Lagrangian equations of motion for mechanical systems with holonomic constraints[14]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q + B^T\lambda \tag{1}$$

$$\Phi(q) = \mathbf{0} \tag{2}$$

where L is the system Lagrangian, Φ are the constraint conditions, q are the generalized coordinate components, λ is the Lagrangian multiplier, Q is the generalized applied force, and $B \in R^{m\times n}$ is the Jacobian matrix of the constraint equations. Equations (1) and (2) can be derived and given as follows:

$$M\ddot{q} + B^T\lambda = F \tag{3}$$

$$\Phi(q) = 0 \tag{4}$$

where $M \in R^n$ denotes the mass matrix of the system and F consists of the applied force, the centrifugal and Coriolis acceleration, and the internal spring force. Because the solution procedure of Eqs. (3) and (4) suffers various drawbacks such as constraint violation in the computer implementation,[8] researchers such as Baumgarte have been motivated to look for alternative solution procedures that can overcome these difficulties. In Baumgarte's technique, one replaces Eq. (4) with

$$\ddot{\Phi} + 2\alpha\dot{\Phi} + \beta^2\Phi = 0 \tag{5}$$

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where

$$\dot{\Phi} = B\dot{q}, \qquad \ddot{\Phi} = B\ddot{q} + \dot{B}\dot{q} \tag{6}$$

and where α and β are both positive real numbers. Substituting Eq. (6) into Eq. (5) and replacing $\ddot{q}$ from the equations of motion, the Lagrange multiplier for holonomic constraints yields the following expression:

$$\lambda = (BM^{-1}B^T)^{-1}(BM^{-1}F + B\dot{q} + 2\alpha\dot{\Phi} + \beta^2\Phi) \tag{7}$$

The constraint forces in Eq. (7) are an output-feedback-like control technique. Notice that an output-feedback control scheme for a dynamic system does not guarantee the stability of the overall system if its internal dynamics (the unobservable part of its dynamics) is unstable. Furthermore, internal stability of the constraint violation stabilization problem has not been taken into consideration by Baumgarte.[7,8] In this regard, we resolve this problem and propose a new stabilization technique by using input–output feedback linearization techniques.

III. Input–Output Feedback Linearization

Input–output feedback linearization is a popular approach to nonlinear control design that has attracted a great deal of interest in recent years.[16–19] The central idea of this approach is to algebraically transform a nonlinear dynamic system into a (fully or partially) linear one so that the linear control techniques can be applied. The standard input–output feedback linearization is given as follows.

Consider a single-input/single-output (SISO) nonlinear dynamic system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8}$$

where $x \in R^n$ is the state variable, $u \in R^1$ the control input, and $y \in R^1$ the system output. Differentiating the second part of Eq. (8) with respect to time once, one obtains

$$\begin{aligned}\dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}f + \frac{\partial h}{\partial x}g \cdot u \\ &= L_f h + L_g h \cdot u\end{aligned} \tag{9}$$

where $L_f h = (\partial h/\partial x) f$ and $L_g h = (\partial h/\partial x) g$ are called the directional (Lie) derivatives of h in the direction of f and g, respectively. Furthermore, the directional derivative has the following property:

$$L_f^k h = L_f\left(L_f^{k-1}h\right) = L_f^{k-1}(L_f h) \tag{10}$$

If $L_g h \neq 0 \;\; \forall x \in U_\varepsilon(x_0)$, where $U_\varepsilon(x_0)$ denotes a ball with its center located on x_0 with radius ε, then let

$$z_1 = h(x), \qquad u = (1/L_g h)[-L_f h + \nu] \tag{11}$$

and we obtain

$$\dot{z}_1 = \nu, \qquad \dot{\eta} = \Lambda(z_1, \eta, \nu), \qquad y = z_1 \tag{12}$$

where η is chosen such that $[z_1^T \;\; \eta^T]^T$ is a diffeomorphism. (A function $\Gamma : R^n \to R^n$, defined in a region Ω, is called a diffeomorphism if it is smooth and if its inverse Γ^{-1} exists and is smooth. A diffeomorphism can be used to transform a nonlinear system into another linear or nonlinear system in terms of a new set of states, similar to what is commonly done in the analysis of linear systems.) If $L_g h = 0$ for some $x \in U_\varepsilon(x_0)$, we differentiate Eq. (9) with respect to time to obtain

$$\begin{aligned}\ddot{y} &= \frac{\mathrm{d}}{\mathrm{d}t}(L_f h) = \frac{\partial}{\partial x}(L_f h)\dot{x} \\ &= \frac{\partial}{\partial x}(L_f h) \cdot (f + gu) \\ &= L_f^2 h + L_g L_f h \cdot u\end{aligned} \tag{13}$$

Again, if $L_g L_f h \neq 0 \;\; \forall x \in U_\varepsilon(x_0)$, then we set

$$\begin{gathered} z_1 = h(x), \qquad z_2 = \dot{z}_1 = L_f h \\ u = \frac{1}{L_g L_f h}\left[-L_f^2 h + \nu\right] \end{gathered} \tag{14}$$

From Eqs. (8–14), we conclude that

$$\begin{gathered} \dot{z}_1 = z_2, \qquad \dot{z}_2 = \nu \\ \dot{\eta} = \Lambda(z_1, z_2, \eta, \nu), \qquad y = z_1 \end{gathered} \tag{15}$$

where η is chosen such that $[z_1^T \;\; z_2^T \;\; \eta^T]^T$ is a diffeomorphism. If we continue the same process until $L_g L_f^k h \neq 0 \;\; \forall x \in U_\varepsilon(x_0)$, the system equations become

$$\begin{gathered} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_k = z_{k+1} \\ \dot{z}_{k+1} = \nu \\ \dot{\eta} = \Lambda(z_1, \ldots, z_{k+1}, \eta, \nu), \qquad y = z_1 \end{gathered} \tag{16}$$

where η is chosen such that $[z_1^T \;\ldots\; z_{k+1}^T \;\; \eta^T]^T$ is a diffeomorphism.

We can apply these derivations to the MBD systems written in terms of DAE as follows.

The equations of motion of an MBD system with holonomic constraints are defined in Eqs. (3) and (4). If we define $x = [q^T \;\; \dot{q}^T]^T$, then Eqs. (3) and (4) can be rewritten as

$$\dot{x} = f(x) + g(x)\lambda, \qquad y = h(x) = \Phi(x) \tag{17}$$

where

$$f = \begin{bmatrix} \dot{q} \\ M^{-1}F \end{bmatrix}, \qquad g = \begin{bmatrix} \mathbf{0}_{n\times m} \\ -M^{-1}B^T \end{bmatrix} \tag{18}$$

where $\mathbf{0}_{n\times m}$ denotes the $(n \times m)$-dimension zero matrix.

Applying the input–output feedback linearization technique given in Eqs. (8–16), we obtain

$$L_g h = [B \quad \mathbf{0}_{m\times n}] \cdot \begin{bmatrix} \mathbf{0}_{n\times m} \\ -M^{-1}B^T \end{bmatrix} = \mathbf{0}_{m\times m} \tag{19}$$

$$L_f h = [B \quad \mathbf{0}_{m\times n}] \cdot \begin{bmatrix} \dot{q} \\ M^{-1}F \end{bmatrix} = B\dot{q} \tag{20}$$

$$L_g L_f h = [\dot{B} \quad B] \cdot \begin{bmatrix} \mathbf{0}_{n\times m} \\ -M^{-1}B^T \end{bmatrix} = -BM^{-1}B^T \tag{21}$$

$$L_f^2 h = [\dot{B} \quad B] \cdot \begin{bmatrix} \dot{q} \\ M^{-1}F \end{bmatrix} = \dot{B}\dot{q} + BM^{-1}F \tag{22}$$

If $BM^{-1}B^T$ is nonsingular, we conclude that

$$\begin{gathered} z_1 = h(x) = y, \qquad z_2 = L_f h = B\dot{q} \\ \lambda = (L_g L_f h)^{-1}\left(-L_f^2 + \nu\right) = (BM^{-1}B^T)^{-1}(\dot{B}\dot{q} + BM^{-1}F - \nu) \end{gathered} \tag{23}$$

which implies that

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \nu, \qquad \dot{\eta} = \Lambda(z_1, z_2, \eta, \nu), \qquad y = z_1 \tag{24}$$

When this input–output feedback linearization technique is applied to DAE, the overall system in Eq. (17) can be divided into two subsystems.

Subsystem I:

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \nu \tag{25}$$

Subsystem II:

$$\dot{\eta} = \Lambda(z_1, z_2, \eta, \nu) \tag{26}$$

Note that the variation of subsystem II does not influence subsystem I. Therefore, if the zero dynamics (defined to be the internal dynamics of the dynamic system when the system output is kept at zero by the input) of subsystem II is stable, i.e.,

$$\dot{\eta} = \Lambda(\mathbf{0}_m, \mathbf{0}_m, \eta, \mathbf{0}_m) \tag{27}$$

is stable (see Appendix), then the DAE given in Eqs. (3) and (4) can be reduced to finding a control law to stabilize the linear subsystem I, i.e.,

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \nu$$

or

$$\dot{z} = Gz + H\nu \tag{28}$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad G = \begin{bmatrix} \mathbf{0}_{m\times m} & \boldsymbol{I}_{m\times m} \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \end{bmatrix}, \qquad H = \begin{bmatrix} \mathbf{0}_{m\times m} \\ \boldsymbol{I}_{m\times m} \end{bmatrix} \tag{29}$$

Because matrices G and H are controllable, the control objective of Eq. (28) can be achieved by developing suitable control inputs. The next section discusses several control techniques for a typical regulation problem that is governed by Eq. (28).

IV. Control Strategy

There are many linear control techniques based on the state-space representation that have been proposed over the last 30 years. The most commonly used techniques are the pole placement technique and the variable structure control (VSC) technique.[19] We will discuss these two control techniques and compare their effectiveness in stabilizing constraint violation by using a numerical example.

A. Pole Placement Technique

The most commonly used technique for a linear control problem is the pole placement technique. In the pole placement technique, the control input is regarded as a linear combination of the states, i.e.,

$$\nu = -Kz \tag{30}$$

The closed-loop system is represented by

$$\dot{z} = (G - HK)z \tag{31}$$

The objective of this control strategy is to find the feedback matrix K such that the real parts of the eigenvalues of $(G - HK)$ are all negative. If we choose a special form of K as $[k_1\boldsymbol{I}_{m\times m} \;\; k_2\boldsymbol{I}_{m\times m}]$, where $k_1, k_2 > 0$, then

$$\det[s\boldsymbol{I} - (G - HK)] = (s^2 + k_2 s + k_1)^m \tag{32}$$

where det( ) denotes the determinant of a matrix. Because k_1, $k_2 > 0$, the real part of the eigenvalues of $(G - HK)$ are all negative. Therefore, if we choose

$$\nu = -k_1 z_1 - k_2 z_2 \tag{33}$$

from Eq. (23), the corresponding Lagrange multiplier is given by

$$\lambda = (BM^{-1}B^T)^{-1}\big(BM^{-1}F + \dot{B}\dot{q} + k_2\dot{\Phi} + k_1\Phi\big) \tag{34}$$

which is referred to as Baumgarte's constraint violation technique.

B. VSC

VSC is a variable high-speed switching feedback control, e.g., the gains in each feedback path switch between two values according to some rules, based on bang-bang control theory. Essentially, VSC utilizes a high-speed control law to drive the plant's state trajectories onto a user-chosen surface (called the sliding or switching surface) in state space and to maintain the plant's trajectories on this surface for all subsequent time. The system dynamics restricted to this surface represents the controlled system's behavior. By proper design of the sliding surface, VSC can be used to attain control objectives such as stabilization, regulation, tracking, and so on.[19–21]

There are two major steps in designing a variable structure controller. The first step is choosing the sliding surface such that the goal of control is achieved, and the second step is designing the control algorithm such that the reaching and sliding conditions are guaranteed. To start with these procedures, we define $\tilde{\Phi} = \Phi - \Phi_d$ to be the constraint error vector and set $\tilde{\Phi} = [\tilde{\phi}_1 \;\; \tilde{\phi}_2 \;\; \cdots \;\; \tilde{\phi}_m]^T$. The problem of stabilizing the constraint violation is equivalent to tracking $\Phi_d = 0$, which yields $\tilde{\Phi} = \Phi = 0$. Let us define a time-varying surface in the state space by the scalar equation

$$s_i(\phi_i, t) = 0 \tag{35}$$

where

$$s_i(\phi_i, t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + c_i\right)^n \cdot \phi_i \tag{36}$$

In vector form

$$\boldsymbol{s}(\Phi, t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + C\right)^n \cdot \Phi \tag{37}$$

where $\boldsymbol{s} = [s_1 \;\; s_2 \;\; \cdots \;\; s_m]^T$ and c_i are strictly positive constants for all i.

The problem of tracking $\tilde{\phi}_i = \phi_i = 0$ is equivalent to that of remaining on the surface $s_i(\phi_i, t) = 0$ for all $t \geq 0$; indeed, $s_i(\phi_i, t) = 0$ represents a linear differential equation whose unique solution is $\phi_i = 0$. Thus, the problem of tracking the constraint error vector can be reduced to that of keeping the quantity s_i at zero for all i (Ref. 19).

Equation (35) can be easily achieved by choosing the control law ν such that

$$\lim_{s_i \to 0} s_i \dot{s}_i < 0, \qquad \forall i = 1, \ldots, m \tag{38}$$

This equation is known as the reaching and sliding condition. In Eq. (36), if we choose $n = 1$, the sliding surface is

$$\boldsymbol{s} = \dot{\Phi} + C\Phi = z_2 + Cz_1 \tag{39}$$

For simplicity, we choose C as $c \cdot \boldsymbol{I}_{m\times m}$, i.e., $c_i = c$ for all i, and then Eq. (37) becomes

$$\boldsymbol{s} = \dot{\Phi} + c\Phi = z_2 + cz_1 \tag{40}$$

To satisfy the reaching and sliding condition (38), a possible control law is

$$\nu = -c\dot{\Phi} - \boldsymbol{W} \cdot \boldsymbol{s} \tag{41}$$

where $\boldsymbol{W}$ is a user-chosen positive definite matrix. Substituting Eq. (41) into Eq. (23), the Lagrange multiplier in this development is given by

$$\lambda = (BM^{-1}B^T)^{-1}(BM^{-1}F + \dot{B}\dot{q} + c\dot{\Phi} + \boldsymbol{W} \cdot \boldsymbol{s}) \tag{42}$$

A similar result could be obtained by using the integral control, i.e., the integral term

$$\int_0^t \Phi(\tau)\,\mathrm{d}\tau$$

as the variable of interest. Then the sliding surface can be chosen as

$$\boldsymbol{s} = \dot{\Phi} + 2c \cdot \Phi + c^2 \cdot \int_0^t \Phi(\tau)\,\mathrm{d}\tau \tag{43}$$

To reach the sliding condition (38), a possible control law can be derived in the following expression:

$$\lambda = (BM^{-1}B^T)^{-1}(BM^{-1}F + \dot{B}\dot{q} + 2c\dot{\Phi} + c^2\Phi + \boldsymbol{W} \cdot \boldsymbol{s}) \tag{44}$$

Note that Eq. (34) is a special case of Eq. (44) with $\boldsymbol{W} = \boldsymbol{0}_{m\times m}$ and that Eq. (42) is also a special form of Eq. (44) with the coefficient of the Φ term vanished.

C. Modified VSC (MVSC) Technique

One can imagine that, if the variable c in Eq. (44) is not a constant but a time-dependent function varied with the constraints Φ, the control term will vanish faster when Φ goes to zero. This idea motivates us to find an appropriate criterion for choosing the time-varying variable $c(t)$. A simple idea is to choose $c(t)$ as $\|\Phi\|_2$ (where $\|\Phi\|_2$ is the 2-norm of Φ). The corresponding Lagrange multiplier is given by

$$\lambda = (BM^{-1}B^T)^{-1}\big(BM^{-1}F + \dot{B}\dot{q} + 2\|\Phi\|_2\dot{\Phi} + \|\Phi\|_2^2\Phi + \boldsymbol{W}\cdot\boldsymbol{s}\big) \tag{45}$$

Another criterion for choosing $c(t)$ is proposed by using the VSC technique. The idea is that the control terms $2c\dot{\Phi} + c^2\Phi + \boldsymbol{W}\cdot\boldsymbol{s}$ must vanish as soon as the plant's trajectory approaches the sliding surface defined in Eq. (43). In other words, if $c(t) = \|\boldsymbol{s}\|_2$ is chosen, the corresponding Lagrange multiplier is given by

$$\lambda = (BM^{-1}B^T)^{-1}\big(BM^{-1}F + \dot{B}\dot{q} + 2\|\boldsymbol{s}\|_2\dot{\Phi} + \|\boldsymbol{s}\|_2^2\Phi + \boldsymbol{W}\cdot\boldsymbol{s}\big) \tag{46}$$

In the next section, two numerical examples are used to illustrate the effectiveness of stabilizing the constraint violation by using Baumgarte's technique and the proposed techniques.

V. Numerical Examples

A. Single-Pendulum Problem

The example is a classical single-pendulum problem whose governing equations of motions are characterized by the following matrices and constraints [see Eqs. (3) and (4) for their definitions]:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}, \qquad \Phi = \begin{Bmatrix} x - l\cos\theta \\ y - l\sin\theta \end{Bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & l\sin\theta \\ 0 & 1 & -l\cos\theta \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} \tag{47}$$

The initial conditions for two different cases are as follows.

Case 1:

$$\begin{array}{lll} x(0) = 1, & y(0) = 0, & \theta(0) = 0 \\ \dot{x}(0) = 0, & \dot{y}(0) = 0, & \dot{\theta}(0) = 0 \end{array} \tag{48}$$

Case 2:

$$\begin{array}{lll} x(0) = 1, & y(0) = 0, & \theta(0) = 0 \\ \dot{x}(0) = 0, & \dot{y}(0) = -2, & \dot{\theta}(0) = -2 \end{array} \tag{49}$$

The Adams–Bashforth third-order (AB-3) integration method is used in the simulations with step size $h = 0.01$ s. The starting procedure of AB-3 is resolved by using Runge–Kutta fourth-order (RK-4) integration.[15] Figures 1 and 2 show the numerical performances of three techniques: 1) the original system without control, 2) Baumgarte's technique [Eq. (34)], and 3) MVSC techniques [Eq. (46)]. We have conducted simulations with different values of constants that are required in Baumgarte's technique and the MVSC technique. The best choice was found to be $k_1 = 2500$ and $k_2 = 100$ for Baumgarte's technique and $c = 200$, $W = 10 \times \boldsymbol{I}$ for the MVSC technique. In case 1, Fig. 1 shows that the 2-norm of the constraint error by using the MVSC technique is about 10 times better than using Baumgarte's technique. In case 2, Fig. 2 shows that the 2-norm of the constraint error by using the MVSC technique is still 10 times better than Baumgarte's technique, and the transient behavior experienced a shorter rising time.

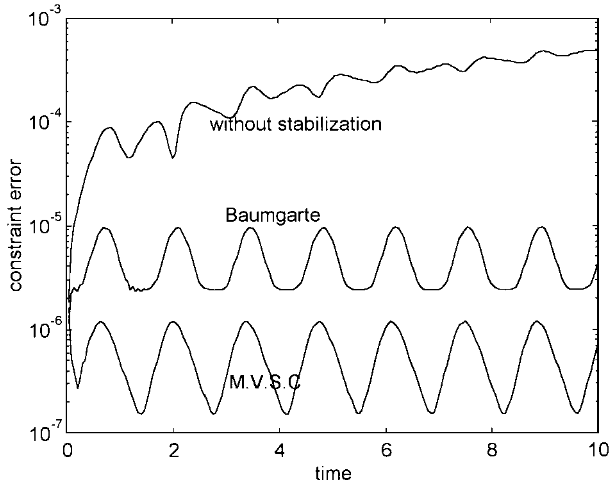

Fig. 1 Constraint errors 2-norm for case 1.

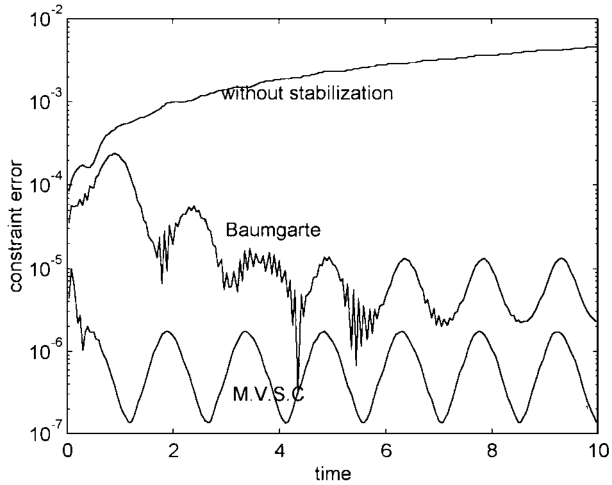

Fig. 2 Constraint errors 2-norm for case 2.

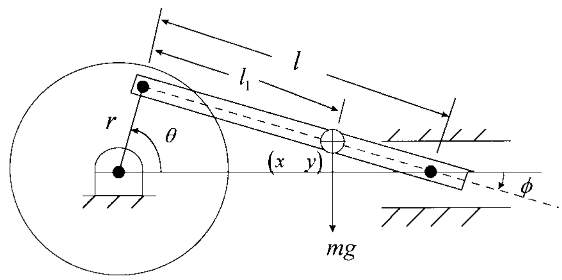

Fig. 3 Crank-slider problem.

B. Crank-Mechanism Problem

The second example is a classical crank mechanism (Fig. 3), whose governing equations of motion are characterized by the following matrices and constraints:

$$M = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & m & \\ & & & m \end{bmatrix} \tag{50}$$

$$\Phi = \begin{Bmatrix} r\cos\theta - (x - l_1\cos\phi) \\ r\sin\theta - (y - l_1\sin\phi) \\ (l - l_1)\sin\phi + y \end{Bmatrix} = 0$$

$$q = [\theta \quad \phi \quad x \quad y]^T, \qquad F = \{0 \quad 0 \quad 0 \quad -mg\}^T \tag{51}$$

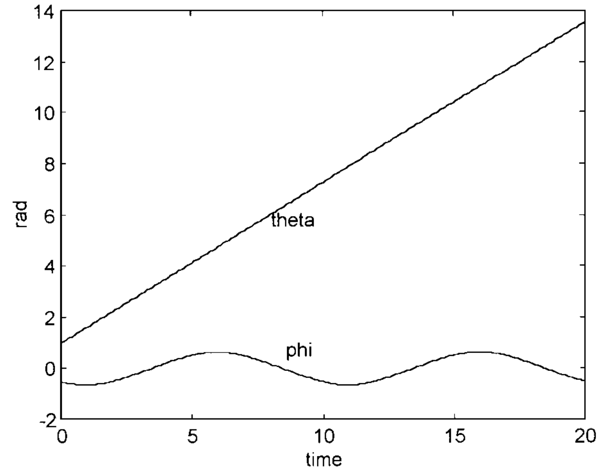

Fig. 4 Plot of θ and ϕ vs time.

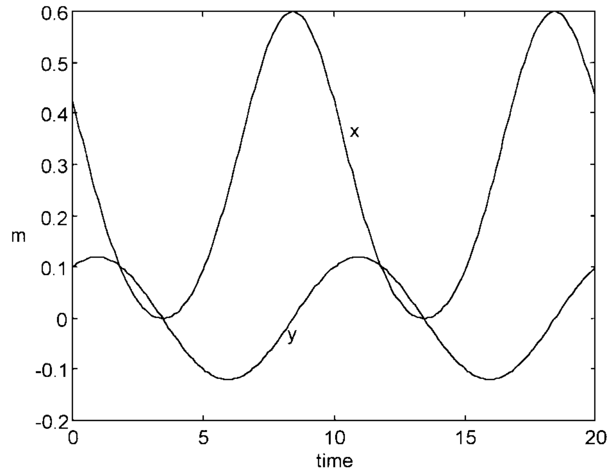

Fig. 5 Plot of x and y vs time.

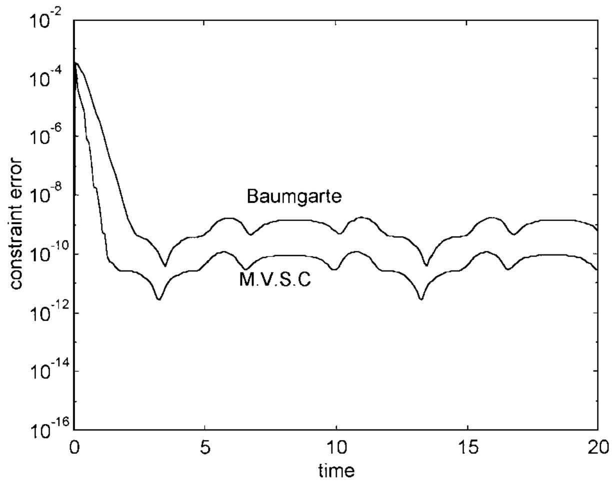

Fig. 6 Constraint errors 2-norm for crank mechanism.

The parameters for the present simulation are $J_1 = 0.045$, $J_2 = 33/4800$, $m = 1$, $r = 0.3$, $l = 0.5$, and $l_1 = 0.3$. The initial conditions are

$$\begin{gathered} \theta(0) = 0.9851, \qquad \phi(0) = -0.5236, \qquad x(0) = 0.4256 \\ y(0) = 0.1, \qquad \dot{\theta}(0) = \pi/5, \qquad \dot{\phi}(0) = -0.240628 \\ \dot{x}(0) = -0.193174, \qquad \dot{y}(0) = 0.041678 \end{gathered} \tag{52}$$

Note that the coefficients that are needed in both Baumgarte's and the MVSC techniques are the same as those given in the first example. The performance of Baumgarte's technique and that of MVSC for this problem are presented in Figs. 4 and 5. We find that the constraint errors by using the MVSC technique are about two orders of magnitude better than with Baumgarte's technique, as shown in Fig. 6.

From the preceding two example problems, we concluded that the MVSC technique gives a more efficient constraint violation stabilization than Baumgarte's technique.

VI. Conclusion

We have presented an input–output feedback linearization technique to transform the nonlinear DAE into a set of linear equations. On reaching the input–output linear relationship with proven stable zero dynamics, the pole placement technique and the VSC technique were applied to the linear system. The former technique led to Baumgarte's technique, whereas the latter technique yielded a new technique for calculating the constraint forces.

From the preliminary numerical experiments reported in Sec. V, we conclude that the proposed technique corrects the constraint violation not only with stability but also with efficiency.

Appendix: Zero Dynamics of DAE

General MBD System

By letting $x = [q^T \;\; \dot{q}^T]$, the DAE form for a general MBD system given in Eqs. (3) and (4) can be written as

$$\dot{x} = f(x) + g(x)\lambda, \qquad y = h(x) = 0 = z_1 \tag{A1}$$

where

$$f(x) = \begin{bmatrix} \dot{q} \\ M^{-1}F \end{bmatrix}, \qquad g(x) = \begin{bmatrix} \mathbf{0}_{n \times m} \\ -M^{-1}B^T \end{bmatrix} \tag{A2}$$

To obtain the unobservable part of the dynamics described by Eq. (27), the nonlinear coordinates transformation technique developed by Isidori[22] is introduced. As developed by Isidori, the unobservable states η can be obtained by a standard procedure to make $[z_1^T \;\; z_2^T \;\; \eta^T]^T$ a diffeomorphism if the distribution $\Delta = \text{span}\{g_1 \cdots g_m\}$ is involutive (Ref. 22 gives a detailed description), where

$$g_i = \begin{bmatrix} \mathbf{0} \\ M^{-1}b_i^T \end{bmatrix}$$

and b_i^T is the ith column of Jacobian matrix B^T.

Proposition 1: The distribution $\Delta = \text{span}\{g_1 \ldots g_m\}$ is involutive.

Proof: Because

$$g_i = \begin{bmatrix} \mathbf{0} \\ M^{-1}b_i^T \end{bmatrix}$$

thus $[\partial(M^{-1}b_i^T)/\partial\dot{q}] = \mathbf{0}$ for all $1 \le i \le m$. The Lie bracket of g_i and g_j is derived as follows:

$$\begin{aligned} [g_i, g_j] &= \frac{\partial g_j}{\partial x} g_i - \frac{\partial g_i}{\partial x} g_j \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\partial\left(M^{-1}b_j^T\right)}{\partial q} & \dfrac{\partial\left(M^{-1}b_j^T\right)}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}b_i^T \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\partial\left(M^{-1}b_i^T\right)}{\partial q} & \dfrac{\partial\left(M^{-1}b_i^T\right)}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}b_j^T \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\partial\left(M^{-1}b_j^T\right)}{\partial q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}b_i^T \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\partial\left(M^{-1}b_i^T\right)}{\partial q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}b_j^T \end{bmatrix} \\ &= \mathbf{0} \end{aligned} \tag{A3}$$

The zero Lie bracket derived in Eq. (A3) implies

$$\operatorname{rank}[g_1(x) \ \cdots \ g_m(x)] = \operatorname{rank}[g_1(x) \ \cdots \ g_m(x) \quad [g_i, g_j](x)] \tag{A4}$$

for all x and $1 \le i, j \le m$. Hence, the distribution $\Delta =$ span $\{g_1 \ \cdots \ g_m\}$ is involutive. QED

Proposition 2: If $\eta = [\eta_1^T \ \ \eta_2^T]^T$, there exist $\eta_1 = A^T M\dot{q}$ and $\eta_2 = q^i$ such that $L_g\eta_1 = \mathbf{0}$, $L_g\eta_2 = \mathbf{0}$ in which A is a null matrix of B and $\boldsymbol{q}^i$ is the vector that contains the chosen independent coordinates of the system.

Proof:

$$\begin{aligned} L_g\eta_1 &= \frac{\partial \eta_1}{\partial x} g = \frac{\partial (A^T M\dot{q})}{\partial x} g \\ &= \begin{bmatrix} \dfrac{\partial (A^T M\dot{q})}{\partial q} & \dfrac{\partial (A^T M\dot{q})}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}B^T \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial (A^T M\dot{q})}{\partial q} & A^T M \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}B^T \end{bmatrix} = A^T B^T = \mathbf{0} \end{aligned} \tag{A5}$$

$$\begin{aligned} L_g\eta_2 &= \frac{\partial \eta_2}{\partial x} g = \frac{\partial q^i}{\partial x} g = \begin{bmatrix} \dfrac{\partial q^i}{\partial q} & \dfrac{\partial q^i}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ (M^{-1}B^T \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial q^i}{\partial q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ M^{-1}B^T \end{bmatrix} = \mathbf{0} \end{aligned} \tag{A6}$$

QED

From the theorems proposed by Isidori[22] and from propositions 1 and 2, we conclude that $\Theta(x) = [h^T(x) \ \ (L_f h)^T(x) \ \ \eta_1^T(x) \ \ \eta_2^T(x)]^T$ is a diffeomorphism. To derive the zero dynamics of DAE, the input ν has to be kept zero. This condition implies that

$$z_1 = \Phi = \mathbf{0}, \qquad z_2 = \dot{\Phi} = \mathbf{0} \tag{A7}$$

According to proposition 2, we have

$$\eta_1 = A^T M\dot{q} = A^T MA\dot{q}^i \tag{A8}$$

which gives the following expression:

$$\dot{q}^i = (A^T MA)^{-1}\eta_1 \tag{A9}$$

Differentiating η_1 and η_2 once with respect to time, we obtain

$$\begin{aligned} \dot{\eta}_1 &= \frac{\mathrm{d}}{\mathrm{d}t}(A^T M\dot{q}) = \dot{A}^T M\dot{q} + A^T M\ddot{q} \\ &= \dot{A}^T MA\dot{q}^i + A^T(F - B^T\lambda) = \dot{A}^T MA\dot{q}^i + A^T F \\ &= \dot{A}^T MA(A^T MA)^{-1}\eta_1 + A^T F \end{aligned} \tag{A10}$$

$$\dot{\eta}_2 = \dot{q}^i = (A^T MA)^{-1}\eta_1 \tag{A11}$$

Equations (A10) and (A11) are the state-space representation of the following second-order differential equation (ODE):

$$A^T MA\ddot{q}^i + A^T M\dot{A}\dot{q}^i = A^T F \tag{A12}$$

which represents an underlying ODE of Eqs. (3) and (4) (Ref. 23). Thus, we conclude that the zero dynamics of DAE and the ODE of the MBD systems are equivalent. If the ODE of the MBD system is stable, the output-feedback-like scheme is also stable. Note that the null matrix A can be represented as a function of independent coordinates if the MBD system is an open-chain system[23] (such as the single-pendulum problem). On the other hand, A must be calculated numerically when a closed-chain MBD system (such as the crank-mechanism problem) is studied. Next, we derive the zero dynamics of DAE for a single-pendulum problem by using proposition 2.

Single-Pendulum Problem

Following the input–output feedback linearization procedures given in Eqs. (9–16), we obtain

$$\begin{aligned} z_1 &= x - l\cos\theta, & z_2 &= y - l\sin\theta \\ z_3 &= \dot{x} + l\dot{\theta}\sin\theta, & z_4 &= \dot{y} - l\dot{\theta}\cos\theta \end{aligned} \tag{A13}$$

To eliminate λ, it is necessary to find η_1 and η_2 and to force

$$L_g\eta_1 = L_g\eta_2 = 0 \tag{A14}$$

Because the null matrix of B is

$$A = [-l\sin\theta \quad l\cos\theta \quad 1]^T$$

the internal states η_1 and η_2 can be obtained by using proposition 2:

$$\eta_1 = -m\dot{x}l\sin\theta + m\dot{y}l\cos\theta + J\dot{\theta}, \qquad \eta_2 = \theta \tag{A15}$$

The Jacobian matrix of Eqs. (A13) and (A15) is given as

$$\frac{\partial z}{\partial x} = \begin{bmatrix} 1 & 0 & l\sin\theta & 0 & 0 & 0 \\ 0 & 1 & -l\sin\theta & 0 & 0 & 0 \\ 0 & 0 & l\cos\theta\cdot\dot{\theta} & 1 & 0 & l\sin\theta \\ 0 & 0 & l\sin\theta\cdot\dot{\theta} & 0 & 1 & -l\cos\theta \\ 0 & 0 & -m\dot{x}l\cos\theta - m\dot{y}l\sin\theta & -ml\sin\theta & ml\cos\theta & J \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{A16}$$

where $z = [z_1 \ \ z_2 \ \ z_3 \ \ z_4 \ \ \eta_1 \ \ \eta_2]^T$, which is nonsingular for all x. Thus, we conclude that Eqs. (A13) and (A15) are a globally nonlinear coordinates transformation of Eq. (A1). Differentiating Eq. (A15) once with respect to time, we get

$$\dot{\eta}_1 = -m\dot{x}\dot{\theta}l\cos\theta - m\dot{y}\dot{\theta}l\sin\theta - mgl\cos\theta, \qquad \dot{\eta}_2 = \dot{\theta} \tag{A17}$$

The zero dynamics of Eq. (A17) is obtained by setting $z_1 = z_2 = z_3 = z_4 = 0$, i.e.,

$$\begin{aligned} x &= l\cos\theta, & y &= l\sin\theta \\ \dot{x} &= -l\sin\theta\cdot\dot{\theta}, & \dot{y} &= l\cos\theta\cdot\dot{\theta} \end{aligned} \tag{A18}$$

Substituting Eq. (A8) into Eq. (A5), we obtain

$$\begin{aligned} \eta_1 &= -m\dot{x}l\sin\theta + J\dot{\theta} + m\dot{y}l\cos\theta \\ &= ml^2\sin^2\theta\cdot\dot{\theta} + J\dot{\theta} + ml^2\cos^2\theta\cdot\dot{\theta} \\ &= (ml^2 + J)\dot{\theta} \end{aligned} \tag{A19}$$

By time differentiation of Eq. (A15) and use of Eqs. (A1), (A18), and (A19), the zero dynamics of the DAE for a single-pendulum problem yields

$$\begin{aligned} \dot{\eta}_1 &= ml\sin\theta\cdot\dot{\theta}^2\cdot l\cos\theta - ml\cos\theta\cdot\dot{\theta}^2\cdot l\sin\theta - mgl\cos\theta \\ &= -mgl\cos\theta = -mgl\cos\eta_2 \end{aligned} \tag{A20}$$

$$\dot{\eta}_2 = \dot{\theta} = \frac{\eta_1}{ml^2 + J}$$

which recovers the ODE form of a single-pendulum system. Therefore, we conclude that the zero dynamics of DAE form of a single-pendulum system is stable.

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